MSc in Economics for Development

Macroeconomics for Development

Week 2 Class

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Consultation hours: Friday, 2-3pm, Weeks 1,3-8 (MT)

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Week 2 Overview: Economic dynamics and growth

1. The first part of today's class will give an overview of simple dynamic systems and phase diagrams:
   - Phase diagrams are used widely in economics to display the evolution of a dynamic system
   - In this class we focus on a very general, simple model with perfect foresight
   - This model highlights three different types of stability:
     - Globally stable
     - Globally unstable
     - Saddle-path stable
   - Economic models are often saddle-path stable, leading to jumps under rational expectations

2. The second part will apply this to the Romer (1990) endogenous R&D model
   - The Romer model extends the Solow model by including endogenous technical change
   - The dynamics of the system can be expressed using the dynamics of Capital...
   - ...and the dynamics of Technology
   - Combining these, the solution to the system depends on three possible parameter values:
   - $\beta + \theta < 1$
   - $\beta + \theta > 1$
   - $\beta + \theta = 1$

3. Conclusion

References:

- Begg, D.K.H., 1982, *The Rational Expectations Revolution in Macroeconomics: Theories and Evidence*, Phillip Allan Publishers, Oxford
  - Section 3.2
- Barro, R., J., and Sala-i-Martin, X., 1995, *Economic Growth*, McGraw Hill
  - Appendix 1.1.3
- Quinn, N., 2011, *Mathematical Economics and Statistics for Development Economists: Lecture Notes*
  - Chapter 6: Economic Dynamics and Chapter 7: Dynamic Systems
- Chiang, A., and Wainwright, K., 2005, *Fundamental Methods of Mathematical Economics*, HcGraw Hill
  - Chapters 14-19
- Simon, C., and Blume, L., 1994, *Mathematics for Economists*, W.W. Norton and Co.
  - Chapters 23-25

1. Dynamic systems and phase diagrams

Phase diagrams are used widely in economics to display the evolution of a dynamic system

What are phase diagrams?

- Phase diagrams are graphical representations of a dynamic system away from its steady state:
  - Graphical: usually in two dimensions, so there are two variables – the state variable (eg capital) and the choice variable (eg consumption)
  - Dynamic: the variables change over time
  - System: the dynamic equations are solved simultaneously
  - Away from steady state: response of the system to a certain starting point, or shocks

When will you see them?

- Growth (Lecture 2): eg Solow-Swan (1956), Romer (1990)
- Dynamic Efficiency (Lecture 7): eg Ramsey

In this class we focus on a very general, simple model with perfect foresight

The Model

- Two equations in two variables:

$$\frac{\partial x}{\partial t} = \dot{x} = ax + by + h$$

$$\dot{y} = cx + dy + k$$

x and *y* are two general variables (we won't define what they are yet).

•Begin by letting a<0, b>0, c<0, d<0 (we will change these assumptions).

•If a, b have opposite signs and c,d have the same sign we ensure the lines cross.

The Steady State

- The point of intersection of two locii, at which both *x* and *y* are unchanging

$$\dot{x} = 0 \qquad \dot{y} = 0$$

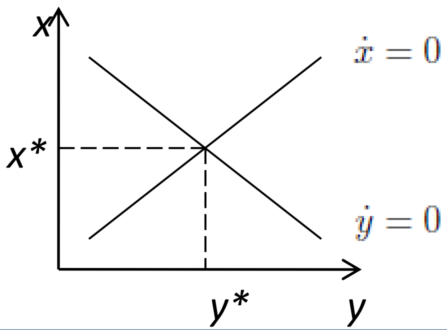
Substitute and rearrange

$$x = \left(\frac{-1}{a}\right)(by + h) \qquad y = \left(\frac{-1}{d}\right)(cx + k)$$

Solve simultaneously

$$x^* = \frac{(bk - dh)}{(ad - bc)} \qquad y^* = \frac{(ch - ak)}{(ad - bc)}$$

The Diagram



We call the lines "locii" as they are a combination of points, for which one variable is unchanging

The phase diagram describes the behaviour of the system away from the steady state

Away from the stationary locus for *x*

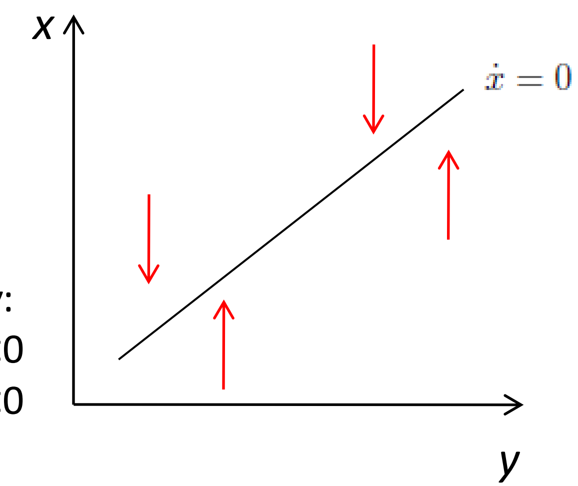
- Equation of motion

$$\dot{x} = ax + by + h$$

- Stationary locus:

$$x = \left(\frac{-1}{a}\right)(by + h)$$

- For a given (*x*,*y*) on this locus, if we make *y*:
  - Larger: $\dot{x} > 0$ as b>0, a<0
  - Smaller: $\dot{x} < 0$ as b>0, a<0



Away from the stationary locus for *y*

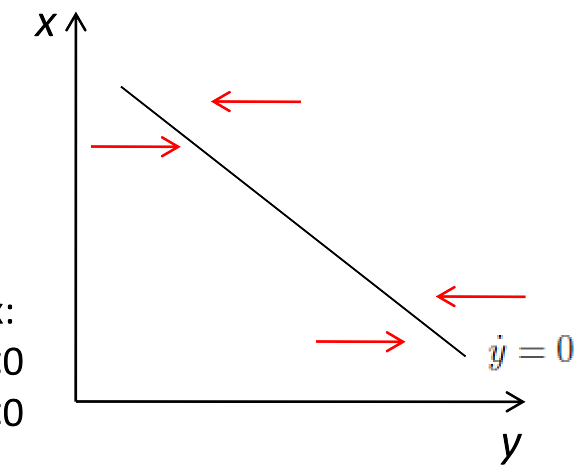
- Equation of motion

$$\dot{y} = cx + dy + k$$

- Stationary locus:

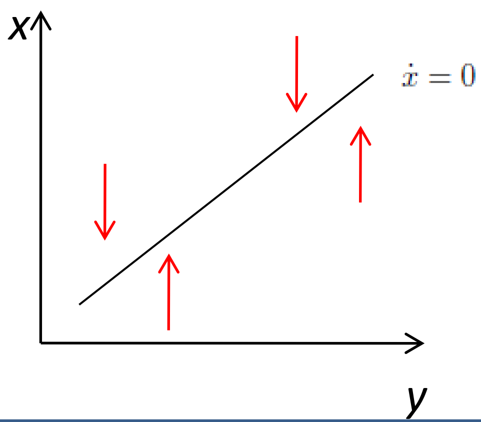
$$y = \left(\frac{-1}{d}\right)(cx + k)$$

- For a given (*x*,*y*) on this locus, if we make x:
  - Larger: $\dot{y} < 0$ as d<0, c<0
  - Smaller: $\dot{y} > 0$ as d<0, c<0

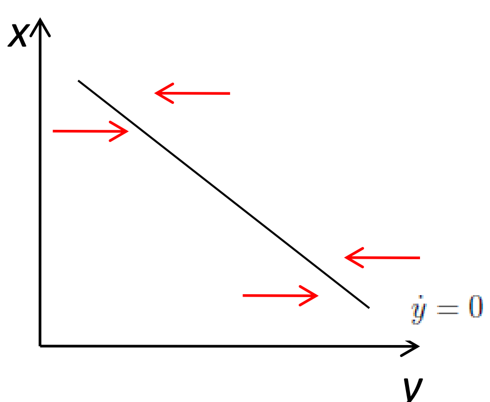


Together these imply movement in both x and y simultaneously away from the locii

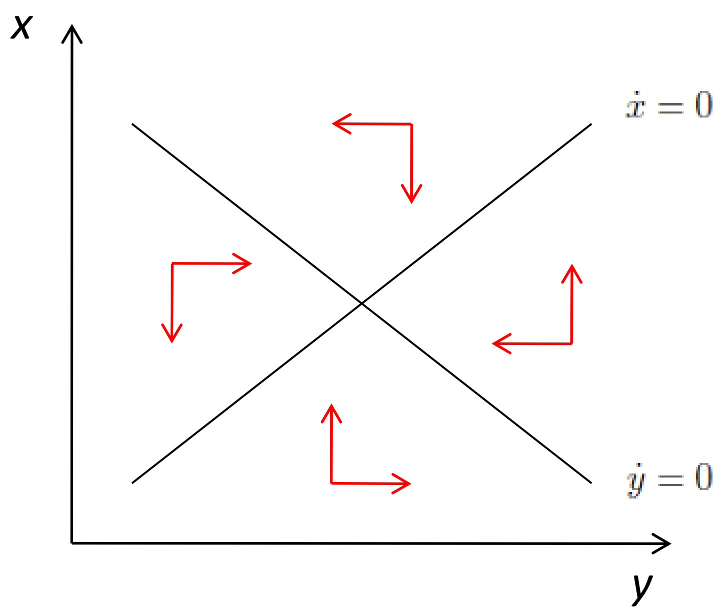
Dynamics of x away from x^dot=0



Dynamics of y away from y^dot=0

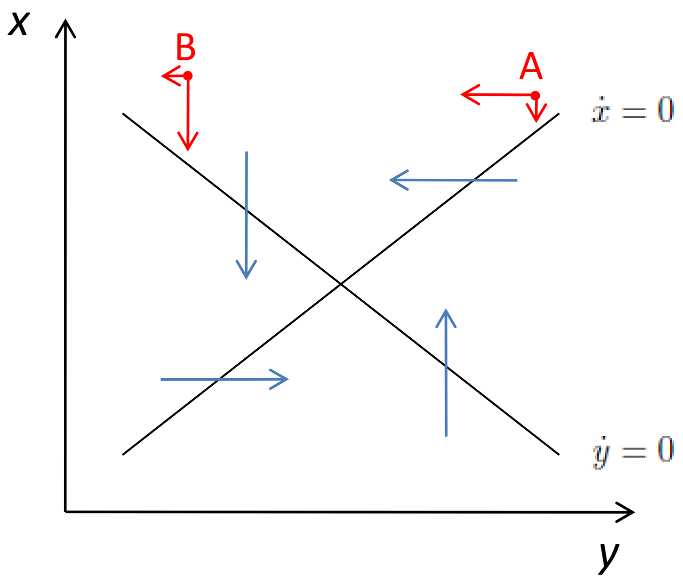


Dynamics of x and y away from the steady state



The speed of adjustment also changes depending on the distance of each variable from the stationary locii

The movement of x and y varies with location



$$\dot{x} = ax + by + h$$

$$\dot{y} = cx + dy + k$$

When parameters:
- a<0
- b>0
- c<0
- d<0

Red arrows

- **A –** close to stationary locus for x:
  - x moves slowly
  - y moves quickly
- **B –** close to stationary locus for y:
  - y moves slowly
  - x moves quickly

Blue arrows

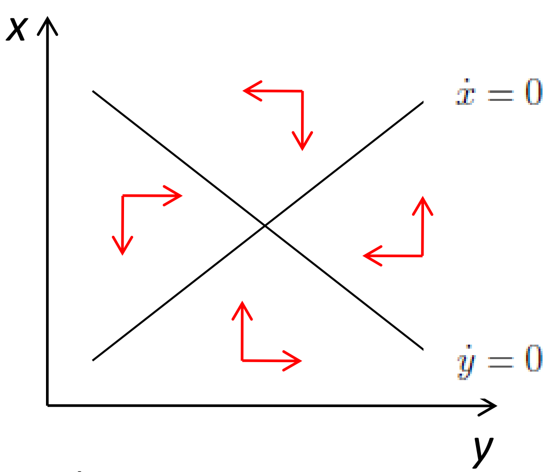
- The system must move horizontally when crossing the stationary locus for x
- The system must move vertically when crossing the stationary locus for y

This model highlights three different types of stability

Equations of motion

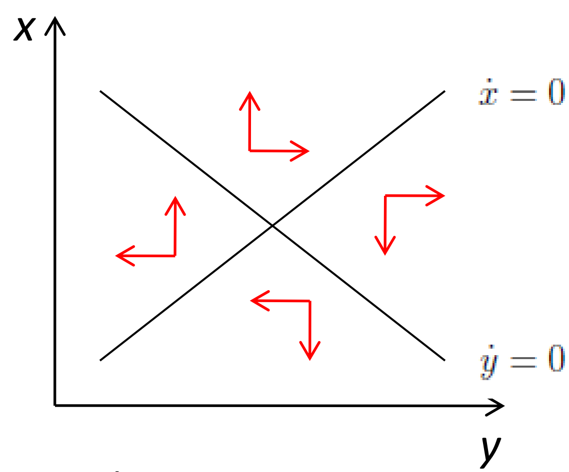
$$\dot{x} = ax + by + h \qquad \dot{y} = cx + dy + k$$

a. Globally Stable



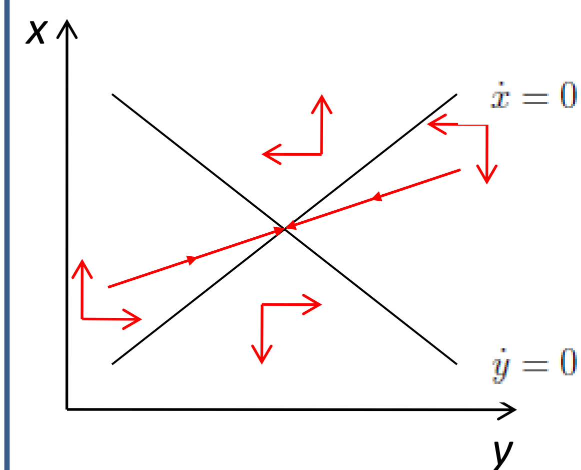
When parameters:
- a<0
- b>0
- c<0
- d<0

b. Globally Unstable



When parameters:
- a>0
- b<0
- c>0
- d>0

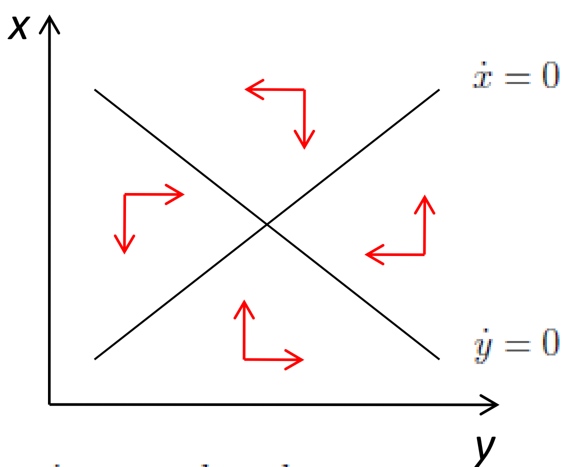
c. Saddle-path Stable



When parameters:
- a>0
- b<0
- c<0
- d<0

This model highlights three different types of stability
a. Globally stable

A globally stable system always reaches the steady state...



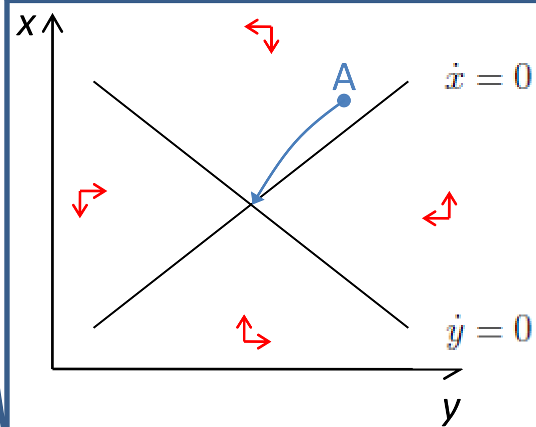
$$\dot{x} = ax + by + h$$

$$\dot{y} = cx + dy + k$$

With parameters:

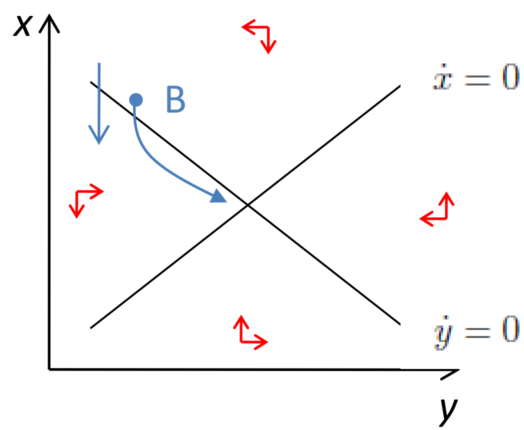
- a<0
- b>0
- c<0
- d<0

...either directly...



x and y change simultaneously

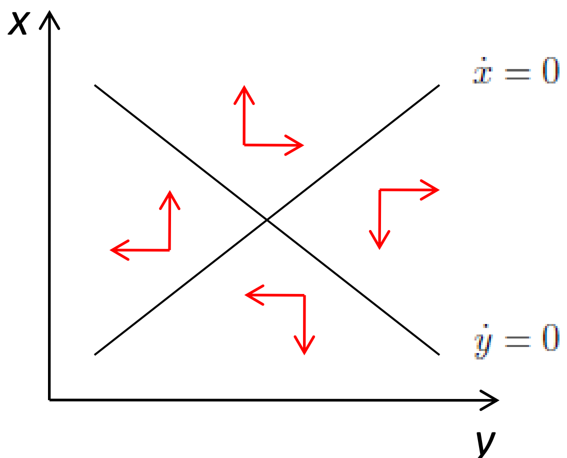
... or after crossing the stationary locii



When crossing the locii, only one variable is changing.

This model highlights three different types of stability
b. Globally unstable

A globally unstable system always moves away from the steady state..



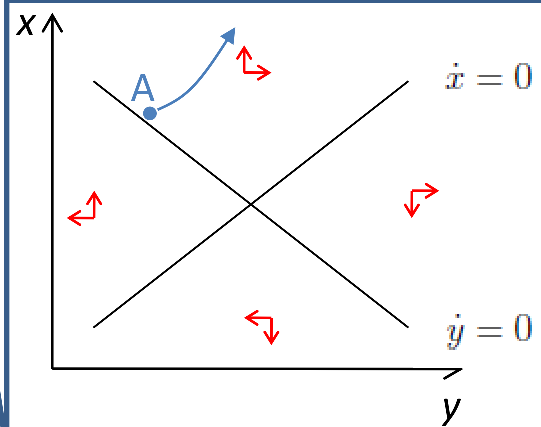
$$\dot{x} = ax + by + h$$

$$\dot{y} = cx + dy + k$$

With parameters:

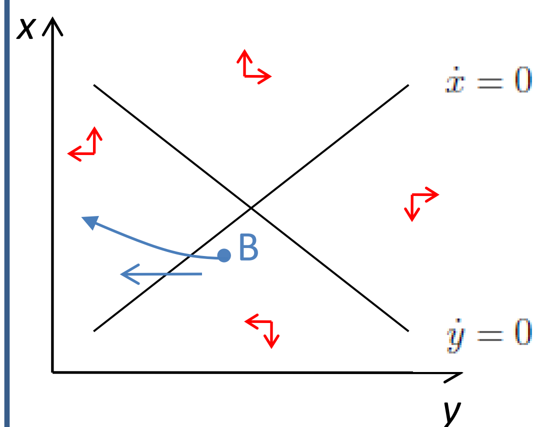
- a>0
- b<0
- c>0
- d>0

...either directly...



x and y change simultaneously

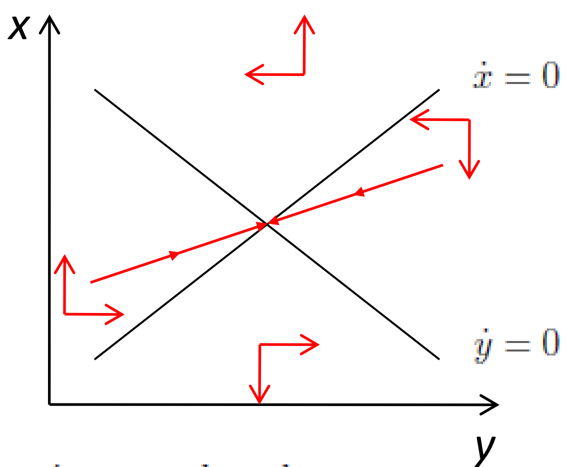
... or after crossing the stationary locii



When crossing the locii, only one variable is changing.

This model highlights three different types of stability
c. Saddle-path stable

A saddle-path stable system depends on the starting point...

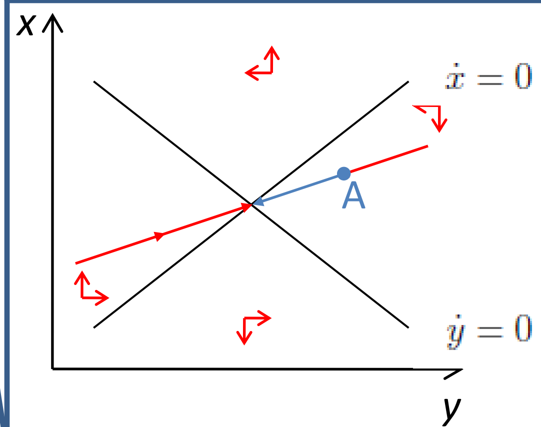


$$\dot{x} = ax + by + h$$

$$\dot{y} = cx + dy + k$$

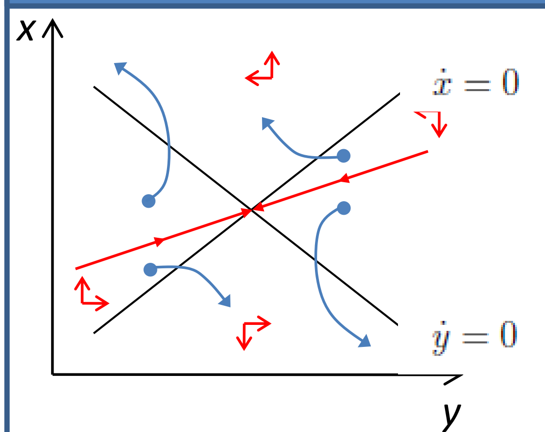
With parameters:
- a>0
- b<0
- c<0
- d<0

...moving to the steady state if on the saddle path



A point in (x,y) space will only converge to the steady state if it is on the unique saddle path

... or moving away from the steady state if off the saddle path



If the starting point is off the saddle path, then the system will diverge.

Economic models are often saddle-path stable, leading to jumps under rational expectations

Economic models are often saddle-path stable	• Economic questions usually involve tradeoffs: •Eg. Consumption vs savings/ production vs investment •Economic answers involve balancing these tradeoffs •The saddle path describes this balance •They often happen by construction – eg transversality conditions.
When expectations are rational this leads to 'jumps' to the saddle path	• Agents with rational expectations know the stable saddle path • If there is a shock to the economy's state variable, the agent chooses the choice variable to be on the saddle path • This leads to jumps (discontinuities) in the choice variable
To solve for the saddle path we use matrix algebra	•Set up the dynamic system of equations in matrix form $\dot{y} = Ay$ •Find eigenvectors and eigenvalues of A •The stable saddle path is given by the eigenvector corresponding to the stable eigenvalue.

Stability of simple First Order Linear Homogenous Ordinary Differential Equations can be analysed using eigenvectors and eigenvalues

- A system of first-order, linear, homogenous ODEs looks like:

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$$

First-order: only $\dot{x}$ not $\ddot{x}$
Linear: no higher order terms
Homogenous: no constant term

- The matrix ***A*** can be decomposed into eigenvectors and eigenvalues

$$\boldsymbol{A}\boldsymbol{V} = \boldsymbol{\Lambda}\boldsymbol{V}$$

- ***A*** describes the evolution of **x**
- ***V*** is a matrix of eigenvectors of ***A***
- ***Λ*** is a diagonal matrix with the eigenvalues of ***A*** on the diagonal

- The eigenvectors ***V*** and eigenvalues ***Λ*** are unique to ***A***. They describe the "pure" directions that x moves in.
  - ***A*** describes the motion of this system as a combination of values of *x* and *y*.
  - However if we express the motion in terms of a different set of axes: ***V*** rather than ***x***,
  - Then, the effect of ***A*** on ***V*** is just a series of scalar movements along ***V***, given by ***Λ***

- We will assume that the eigenvalues ***Λ*** are distinct: so that the eigenvectors ***V*** are linearly independent and thus span the (x,y) space, and the inverse $\boldsymbol{V}^{-1}$ exists.

We can use the properties of eigenvectors and eigenvalues to solve the system of FOLH-ODEs

General Solution If V is the matrix of normalised eigenvectors of A, then

$$\mathrm{V}^{-1}\mathrm{AV} = \Lambda$$

where Λ is a diagonal matrix whose diagonal elements are the eigenvalues of A.[1] Right-multiply both sides by V^{-1} to obtain $\mathrm{V}^{-1}\mathrm{A} = \Lambda \mathrm{V}^{-1}$.

Now consider the system of differential equations, $\dot{\mathbf{x}} = \mathrm{A}\mathbf{x}$. Left-multiplying by V^{-1} yields $\mathrm{V}^{-1}\dot{\mathbf{x}} = \mathrm{V}^{-1}\mathrm{A}\mathbf{x}$ and we can substitute $\Lambda\mathrm{V}^{-1}$ for $\mathrm{V}^{-1}\mathrm{A}$ to obtain

$$\mathrm{V}^{-1}\dot{\mathbf{x}} = \Lambda \mathrm{V}^{-1}\mathbf{x}.$$

But $\mathrm{V}^{-1}\mathbf{x}$ is a vector, call it $\mathbf{z}$, so $\mathrm{V}^{-1}\dot{\mathbf{x}} = \dot{\mathbf{z}}$, and we can write the system as

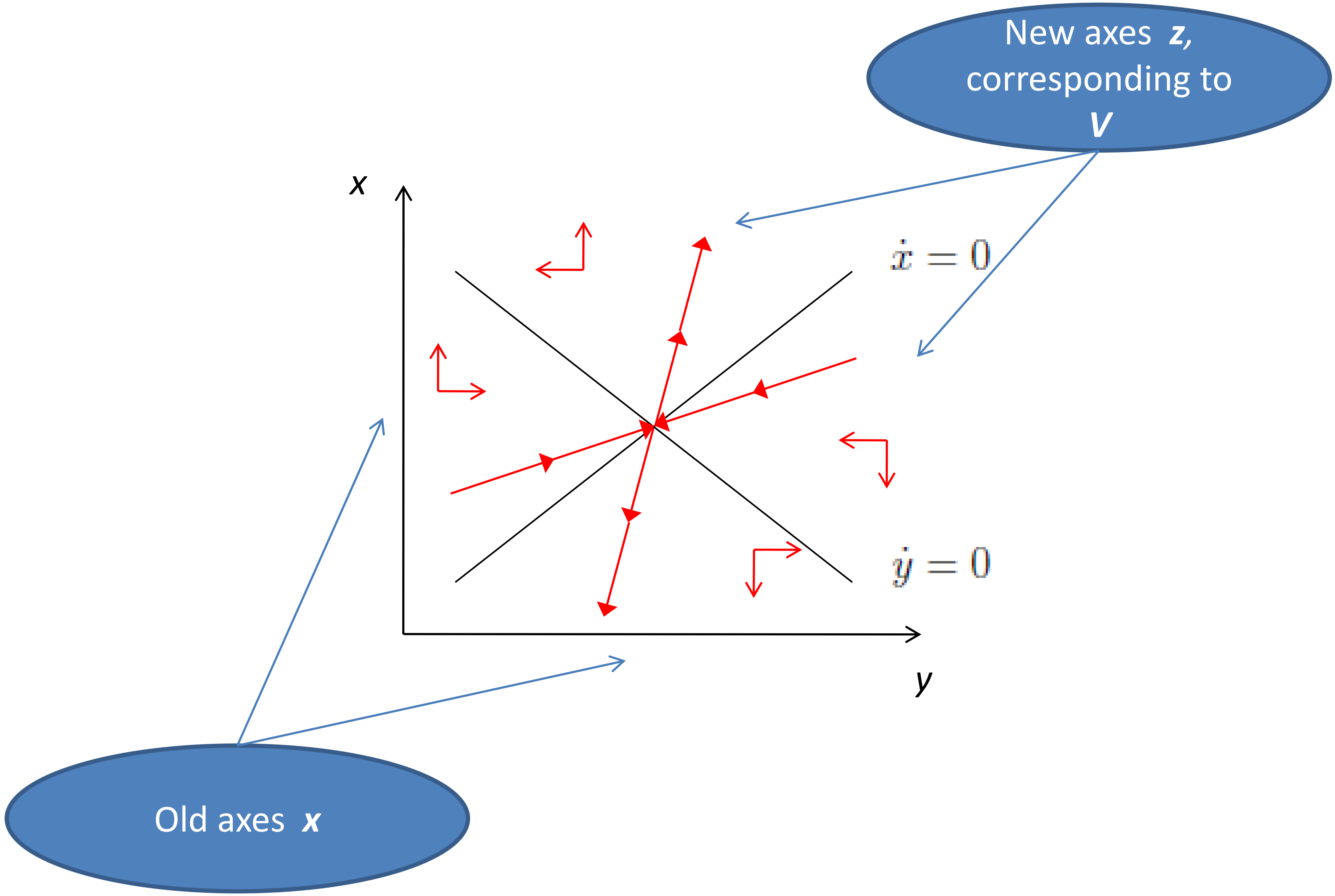
$$\dot{\mathbf{z}} = \Lambda \mathbf{z}.$$

Because Λ is a diagonal matrix, this system of equations is very simple: $\dot{z}_1 = \lambda_1 z_1, \dot{z}_2 = \lambda_2 z_2, \ldots, \dot{z}_n = \lambda_n z_n$, each of which solves to give $z_i = c_i e^{\lambda_i t}$. There are n arbitrary constants $c_1, c_2, \ldots, c_n$; to determine these we would require n boundary conditions. (Knowledge of the initial state of the system $\mathbf{x}(t=0)$ would be sufficient; the n components of this vector provide the n boundary conditions.)

It is now easy to find the solutions $x_i(t)$, as $\mathbf{x} = \mathrm{V}\mathbf{z}$, so each x_i is a linear combination of $z_1(t), z_2(t), \ldots, z_n(t)$.

Source: Quinn, N., 2011, Math Course Lecture Notes: Chapter 7.3

This can be represented using a phase diagram



To find the eigenvalues and eigenvectors we use the roots to the characteristic equation

- The n eigenvalues $\boldsymbol{\lambda_i}$ and eigenvectors $\boldsymbol{v_i}$ of A satisfy:

$$\mathrm{A}\mathbf{v} = \lambda\mathbf{v} \qquad \lambda \in \mathbb{R} \quad \text{is a scalar}$$

- Rearranging:

$$\mathrm{A}\mathbf{v} - \lambda\mathbf{v} = 0 \qquad \mathrm{A}\mathbf{v} - \lambda\mathrm{I}_n\mathbf{v} = (\mathrm{A} - \lambda\mathrm{I}_n)\mathbf{v} = 0 \qquad (*)$$

- This will only have non-trivial solutions (ie ***v***<>***0***) if:

$$|\mathrm{A} - \lambda\mathrm{I}_n| = 0$$

- For the 2x2 case, this determinant gives:

$$(a_{11} - \lambda)(a_{22} - \lambda) - (a_{21}a_{12}) = 0$$

$$\lambda^2 - \lambda(a_{11} + a_{22}) + (a_{11}\,a_{22} - a_{12}a_{21}) = 0 \qquad \text{(characteristic eq'n)}$$

$$\lambda = \frac{(a_{11} + a_{22})}{2} \pm \frac{\sqrt{(a_{11} + a_{22})^2 - 4(a_{11}\,a_{22} - a_{12}a_{21})}}{2}$$

- Substitute each λ into (*) to get the associated eigenvector

This illustrates other types of dynamics: real vs complex eigenvalues (roots of characteristic equation)

Real eigenvalues	Complex eigenvalues
Stable: $\lambda_1<0$ $\lambda_2<0$	Stable Oscillations: Real part of $\lambda_1<0$ $\lambda_2<0$
Unstable: $\lambda_1>0$ $\lambda_2>0$	Unstable Oscillations: Real part of $\lambda_1>0$ $\lambda_2>0$
Saddle-path stable: $\lambda_1<0$ $\lambda_2>0$ (or vice versa)	Repeating Oscillations: no real part of λ_1 λ_2

2. The Romer (1990) model of endogenous growth

The Solow-Swan model was discussed in the lecture. There have been a wide range of extensions to it.

Solow-Swan Limitation	Extension
1. Constant savings rate	• Endogenise savings/consumption (Ramsey, 1928): • Using technological change (Lucas, 1988) • Using capital accumulation (Jones and Manuelli, 1990)
2. Omitted factors esp. human capital ("Lucas Paradox", Lucas, AER, 1990)	• Include human capital: • As schooling (Mankiw, Romer, Weil) • As knowledge (Romer, 1986) • Y=AK model (Romer, 1986) • As skill (Lucas, 1988) • As learning by doing (Young, 1999)
3. Exogenous technical change	•Endogenise technical change: • Using R&D (Romer, 1990) — Will expand in detail • Using Schumpeterian competition (Grossman and Helpman, 1991) • Using both R&D and Schumpeterian competition (Aghion and Howitt, 1992)
4. Aggregation production function assumes optimal resource allocation across economy	• Develop non-aggregate theory (Solow, 2005)

The Romer (1990) model extends the Solow model by including endogenous technical change

Romer (1990) system of equations

Equation	Definition	Assumptions
$Y(t) = [(1-a_K)K(t)]^{\alpha}[A(t)(1-a_L)L(t)]^{1-\alpha}$	Production function	•Constant returns to scale •Diminishing marginal returns •Inada conditions •A proportion of labour and capital is used to produce technology via R&D, rather than output
$\dot{K}(t) = sY(t)$	Dynamics of capital	•Constant savings •No depreciation for simplicity
$\dot{L}(t) = nL(t)$	Dynamics of labour	•Exogenous labour growth
$\dot{A}(t) = B[a_K K(t)]^{\beta}[a_L L(t)]^{\gamma} A(t)^{\theta}$	Dynamics of technology	•Endogenous technology growth via R&D •Capital, labour and the existing technology stock (standing on the shoulders of giants) all contribute to technology growth •Returns to these factors are not necessarily constant •Marginal returns may be increasing (eg for *A*)

The production function shows how the factors are combined and the dynamics show how they evolve over time

Source: Romer (1996), Chapter 3

The dynamics of the system can be expressed using the dynamics of Capital...

We first manipulate the dynamics of capital to express as a growth rate

1. Start with the dynamics of the level of capital

$$\dot{K}(t) = sY(t)$$

2. Express in terms of the growth rate
   - This is because we are looking for the steady state, which is a constant growth rate

$$g_K(t) = \frac{\dot{K}(t)}{K(t)} = c_K \left[\frac{A(t)L(t)}{K(t)}\right]^{1-\alpha}$$

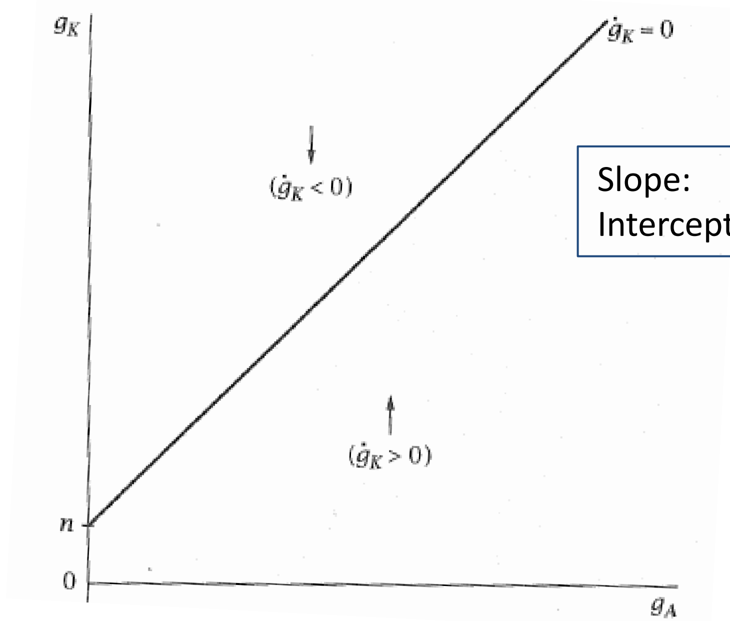
3. Express in terms of the dynamics of the growth rate
   - When the growth rate stops changing, the economy is at the steady state

$$\frac{\dot{g}_K(t)}{g_K(t)} = (1-\alpha)[g_A(t) + n - g_K(t)]$$

We look at $g_K(t)$ rather than $k(t)$ in the Solow-Swan model as the purpose is to separate A and K

The capital dynamics can then be shown diagrammatically

Romer (1990) phase diagram



Slope: *1*
Intercept: *n*

Why do we plot a diagram on these axes?

- The Romer (1990) model is a system of two differential equations, for *A* and *K*
- The steady state is when both *A* and *K* stop moving, or when both growth rates are zero
- Thus, we want to solve simultaneously:

$$\dot{g}_K(t) = 0 \qquad \dot{g}_A(t) = 0$$

...and the dynamics of Technology

We first manipulate the dynamics of capital to express as a growth rate

1. Start with the dynamics of the level of technology

$$\dot{A}(t) = B[a_K K(t)]^{\beta}[a_L L(t)]^{\gamma} A(t)^{\theta}$$

2. Express in terms of the growth rate
   - This is because we are looking for the steady state, which is a constant growth rate

$$g_A(t) = \frac{\dot{A}(t)}{A(t)}$$

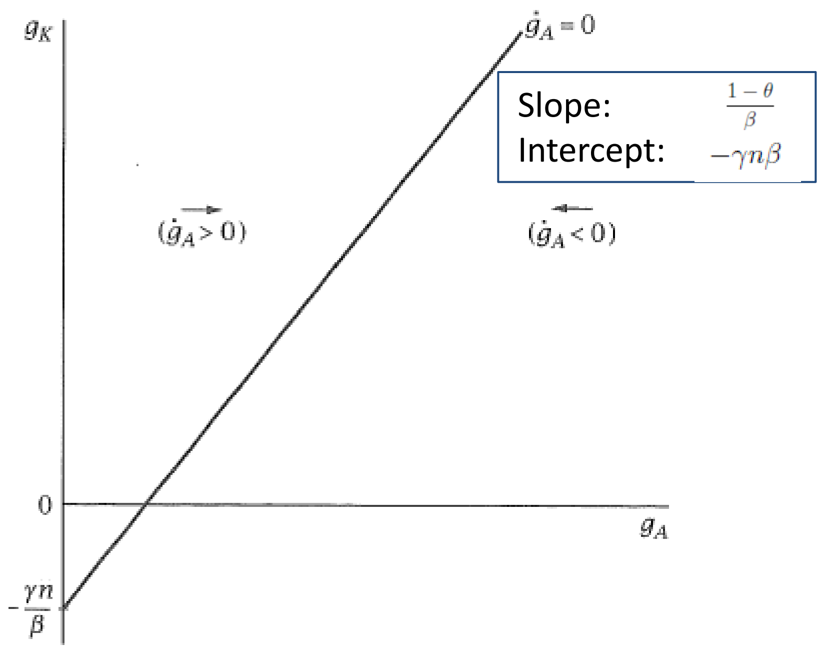
$$= c_A K(t)^{\beta} L(t)^{\gamma} A(t)^{\theta - 1}$$

3. Express in terms of the dynamics of the growth rate
   - When the growth rate stops changing, the economy is at the steady state

$$\frac{\dot{g}_A(t)}{g_A(t)} = \beta g_K(t) + \gamma n + (\theta - 1) g_A(t)$$

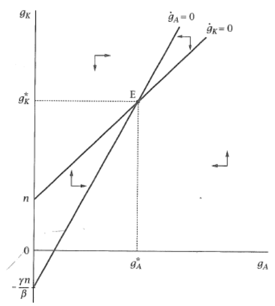
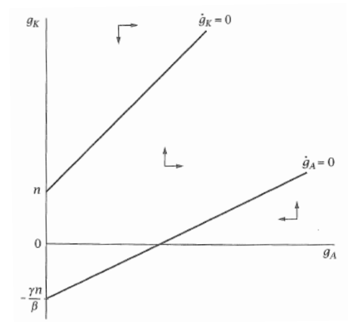
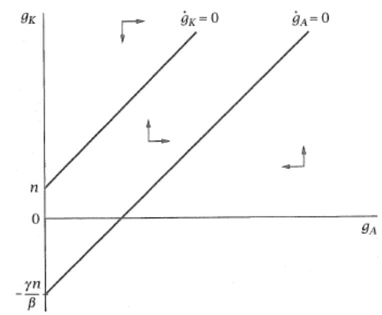
The capital dynamics can then be shown diagrammatically

Romer (1990) phase diagram



Combining these, the solution to the system depends on three possible parameter values:

We don't know how the lines intersect:

Case 1: β + θ <1	Case 2: β + θ >1	Case 3: β + θ =1
		

To find out we look at the slopes of each line, and the contribution of *A* and *K* to the evolution of technology

- The slopes of each line are:

$\dot{g}_K(t) = 0$ *Slope: 1* $\dot{g}_A(t) = 0$ *Slope:* $\frac{1-\theta}{\beta}$

- The evolution of output has constant returns to scale for *A* and *K*

$$Y(t) = [(1-a_K)K(t)]^{\alpha}[A(t)(1-a_L)L(t)]^{1-\alpha}$$

- Thus the stability of the system depends on whether the evolution of technology has decreasing, constant or increasing returns to scale for *A* and *K*

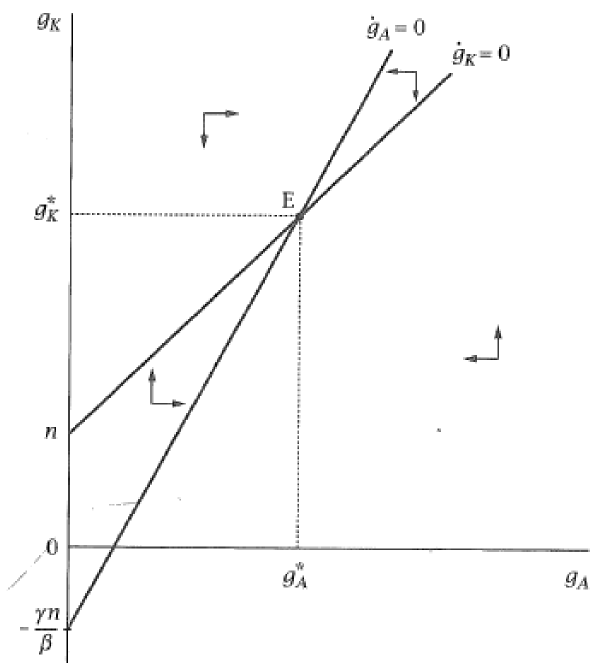
$$\dot{A}(t) = B[a_K K(t)]^{\beta}[a_L L(t)]^{\gamma}A(t)^{\theta}$$

Increasing both *K* and *A* by a factor of *X* increases $\dot{A}(t)$ by a factor of $X^{\beta+\vartheta}$

Case 1: β + θ <1

In this case (1- θ)/ β >1 and the lines intersect

Romer (1990) phase diagram



Equations of Motion

$$\frac{\dot{g}_K(t)}{g_K(t)} = (1-\alpha)[g_A(t) + n - g_K(t)]$$

$$\frac{\dot{g}_A(t)}{g_A(t)} = \beta g_K(t) + \gamma n + (\theta - 1) g_A(t)$$

Thus there will be a steady state growth rate for capital, technology and output

- If β + θ <1 then there are decreasing returns to scale in the production of *A* from adding an extra unit of *A* and *K*
- This induces stability in the system
- The stable steady state is found by simultaneously solving the equations of motion when:

$$\dot{g}_K(t) = \dot{g}_A(t) = 0$$

- The steady state rate of technology growth is then:

$$g_A^* = \frac{\beta + \gamma}{1 - (\theta + \beta)} n$$

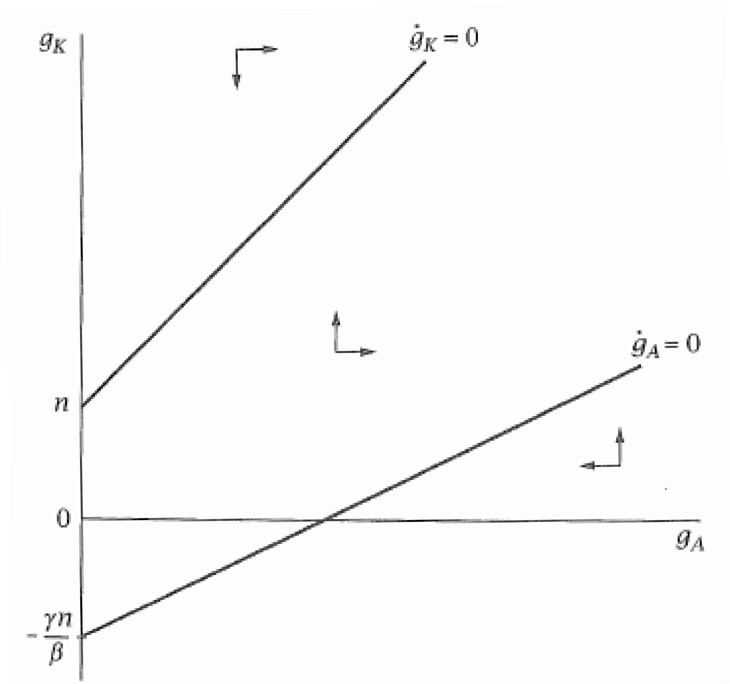
- The steady state rate of growth depends on:
    - population growth
    - returns in production of technology
- The steady state rate of growth doesn't depend on a_L or a_K, shares of labour and capital in R&D sector.
    - These affect the level of technology, but not the growth rate

Romer (1990) phase diagram

Case 2: β + θ >1

In this case (1- θ)/ β <1 and the lines diverge

Romer (1990) phase diagram



Equations of Motion

$$\frac{\dot{g}_K(t)}{g_K(t)} = (1-\alpha)[g_A(t) + n - g_K(t)]$$

$$\frac{\dot{g}_A(t)}{g_A(t)} = \beta g_K(t) + \gamma n + (\theta - 1) g_A(t)$$

Romer (1990) phase diagram

Thus there is no steady state level of growth and output will continue to grow forever

• Wherever the staring point is, eventually the economy will end up in the region between the two lines

•Between the lines both capital and technology, and in turn output, will grow forever

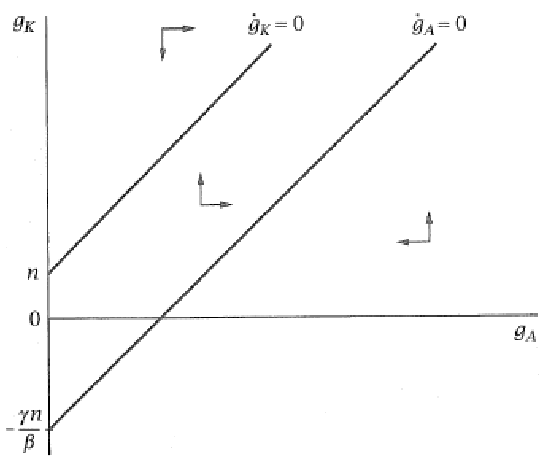
•This happens as there are increasing returns to scale when adding an extra unit of *A* or *K* to the production of technology. This allows the economy to grow in perpetuity

Case 3: β + θ =1

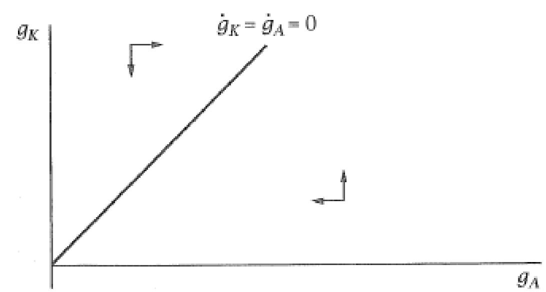
In this case (1- θ)/ β =1 and the lines are parallel

Romer (1990) phase diagram

When n<> 0



When n = 0



Equations of Motion

$$\frac{\dot{g}_K(t)}{g_K(t)} = (1-\alpha)[g_A(t) + n - g_K(t)]$$

$$\frac{\dot{g}_A(t)}{g_A(t)} = \beta g_K(t) + \gamma n + (\theta - 1) g_A(t)$$

There will be balanced growth if the population is constant

- When the population is growing, there are increasing returns to the production of technology and growth is unstable
- When the population is constant the production of technology treads a stable "knife edge" and there will be a balanced growth path.
- We have endogenised (somewhat) the rate of technological growth in the economy.
  - This has pushed back the "frontier of exogeneity" a little
  - We still don't know what causes an economy to invest in more/less R&D

Conclusion

1. The first part of today's class will give an overview of simple dynamic systems and phase diagrams:
   - Phase diagrams are used widely in economics to display the evolution of a dynamic system
   - In this class we focus on a very general, simple model with perfect foresight
   - This model highlights three different types of stability:
     - Globally stable
     - Globally unstable
     - Saddle-path stable
   - Economic models are often saddle-path stable, leading to jumps under rational expectations

2. The second part will apply this to the Romer (1990) endogenous R&D model
   - The Romer model extends the Solow model by including endogenous technical change
   - The dynamics of the system can be expressed using the dynamics of Capital...
   - ...and the dynamics of Technology
   - Combining these, the solution to the system depends on three possible parameter values:
   - $\beta + \theta < 1$
   - $\beta + \theta > 1$
   - $\beta + \theta = 1$

3. Conclusion